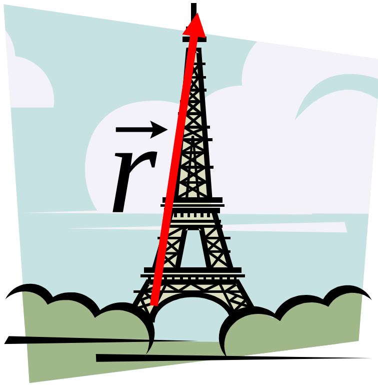
RIGID BODY

A solid body in which deformation is zero or so small it can be neglected

(The distance between any two given points on a rigid body remains constant in time regardless of external forces exerted on it.)



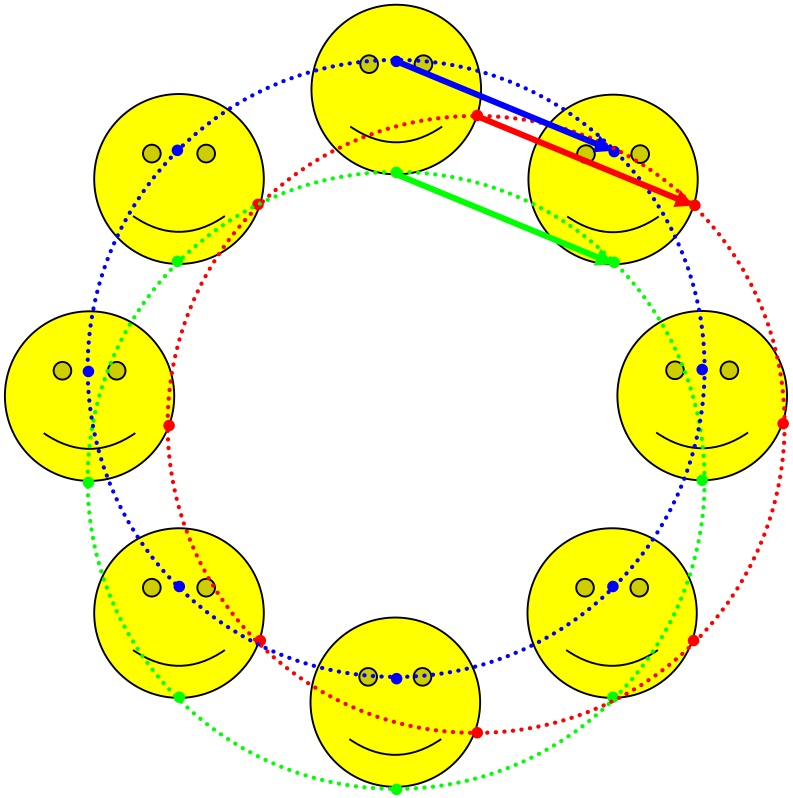
$$\vec{r}_{ij} = \text{const}$$

+ chair
+ bridge

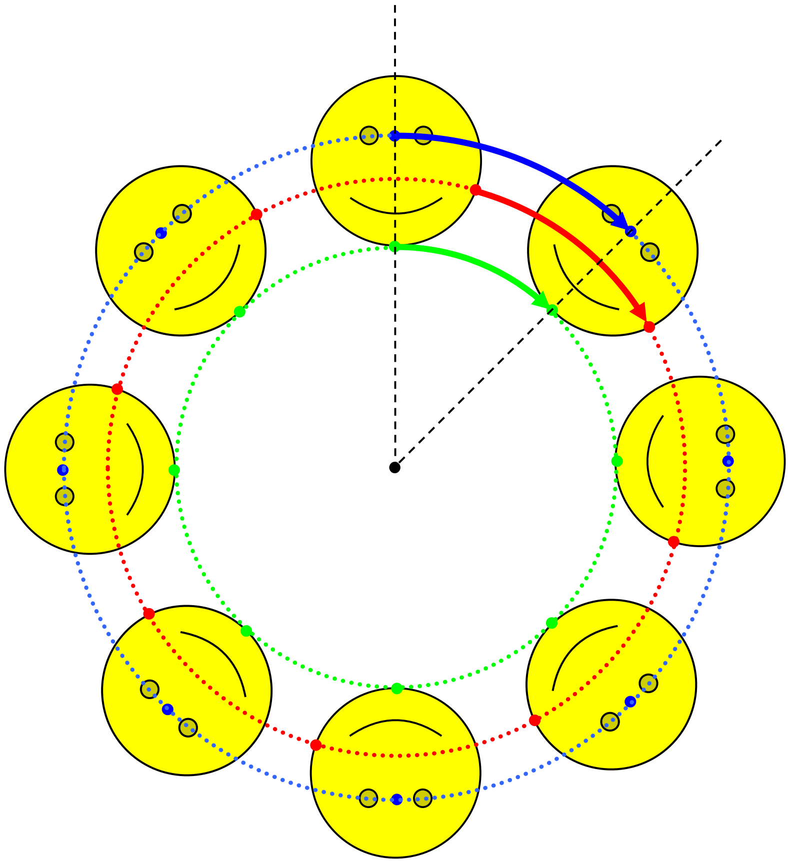
– board sponge
– bridge that resonates

TYPES OF RIGID BODY MOVEMENTS

1) Translational movement in a circle:



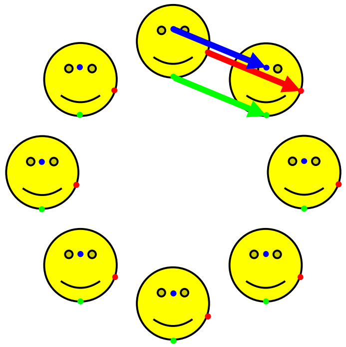
2) Circular motion:



TYPES OF RIGID BODY MOVEMENTS

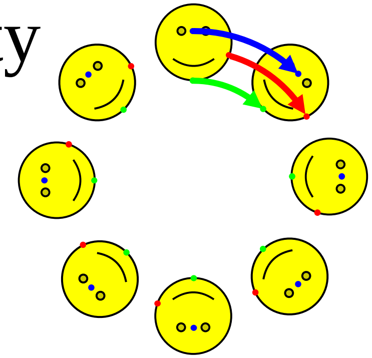
1) **Translational movement**:

All rigid body points are moved in parallel directions through equal distances, with the same speed and acceleration.

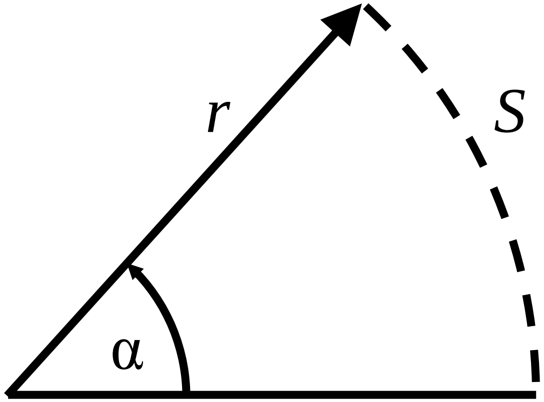


2) **Circular motion**:

All rigid body points rotate along a circular paths with centres lying on the axis of rotation, with the same change of the angular position, the same angular velocity and angular acceleration.



THE ROTATIONAL VARIABLES



- Angular position α

$$\alpha = \frac{S}{r}$$ (radian measure)

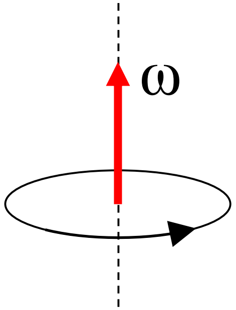
- Angular velocity ω

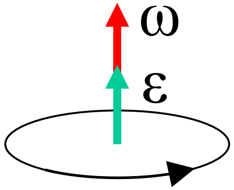
$$\omega = \frac{\mathrm{d}\alpha}{\mathrm{d}t}$$

- Angular acceleration ε

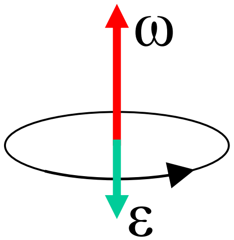
$$\varepsilon = \frac{\mathrm{d}\omega}{\mathrm{d}t}$$

Angular velocity direction is determined by right-hand rule:





Accelerated circular motion



Decelerated circular motion

Equations of motion for constant angular acceleration:

$$\begin{cases} \vec{\alpha}(t) = \vec{\alpha}_0 + \vec{\omega}_0 t + \frac{1}{2}\vec{\varepsilon} t^2 \\ \vec{\omega}(t) = \vec{\omega}_0 + \vec{\varepsilon} t \\ \vec{\varepsilon}(t) = \text{const} \end{cases}$$

$\vec{\alpha}$ – angular position

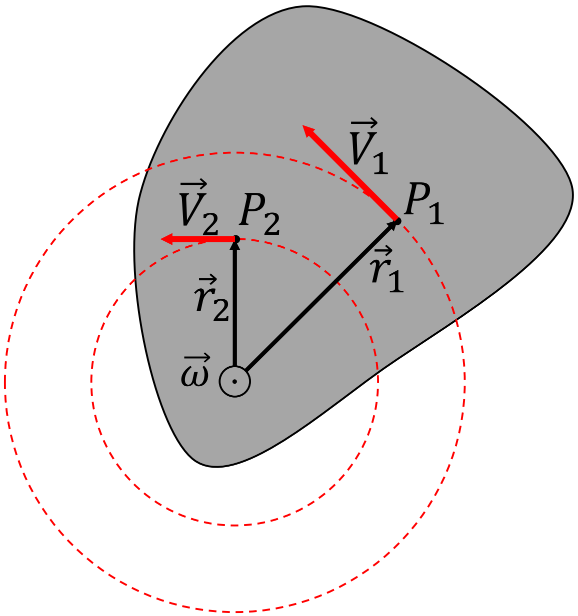
$\vec{\omega}$ – angular velocity

$\vec{\varepsilon}$ – angular acceleration

RELATING THE LINEAR AND ANGULAR VELOCITY

$$S = \alpha r \Longrightarrow \frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}\alpha}{\mathrm{d}t} r \quad (r = \text{const})$$

$$V = \omega r$$



1) $r_2 < r_1 \Longrightarrow V_2 < V_1$

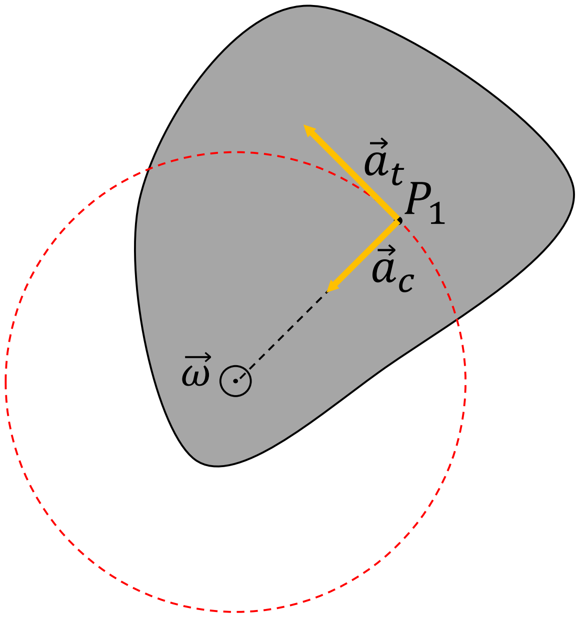
2) $2\pi r = VT \Longrightarrow T = \frac{2\pi r}{V} = \frac{2\pi r}{\omega r} = \frac{2\pi}{\omega} \Longrightarrow \omega = \frac{2\pi}{T}$

RELATING THE LINEAR AND ANGULAR ACCELERATION

$$\mathrm{V} = \omega r \implies \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}t} r \quad (r = \mathrm{const})$$

$$a_t = \varepsilon r$$

$$a_c = \frac{V^2}{r} \implies a_c = \omega^2 r$$

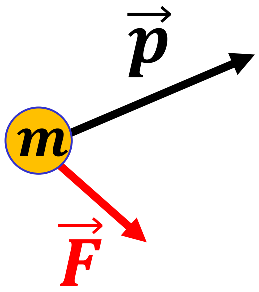


THE BASIC QUANTITIES IN THE LINEAR MOTION

1) **mass – m** (bring about the **Inertia** – the resistance of any physical object to any change in its state of motion)

2) **momentum – $\vec{p} = m\vec{V}$** (describe the state of motion)

3) **force – $\vec{F}$** (the quantity changing the state of motion)



THE BASIC QUANTITIES IN THE CIRCULAR MOTION

1) **moment of inertia** – I (rotational inertia)

2) **angular momentum** – $\vec{L}$ (describe the state of the circular motion)

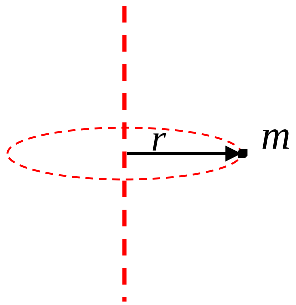
3) **torque** – $\vec{\tau}$ (the quantity changing the state of the circular motion)

MOMENT OF INERTIA

Bring about the **Inertia** – the resistance of any physical object to any change in its state of the circular motion, determined by the mass distribution around the axis of rotation.

For a mass m moving around the rotation axis on a circular path with radius r:

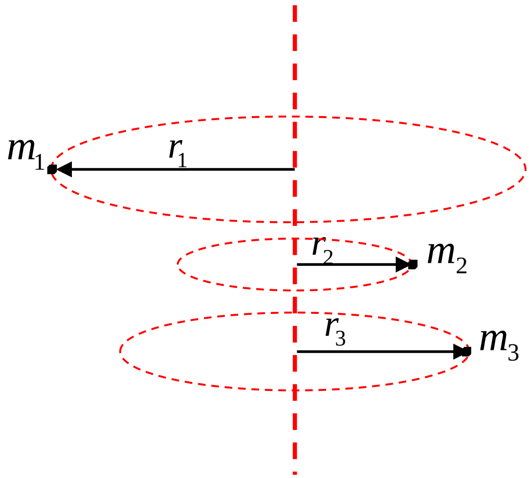
$$I = mr^2$$



MOMENT OF INERTIA

For a system of n masses the total moment of inertia is the sum of moment of inertia for individual points:

$$I = \sum_{i=1}^{n} I_i = \sum_{i=1}^{n} m_i r_i^2$$



MOMENT OF INERTIA

For rigid body:

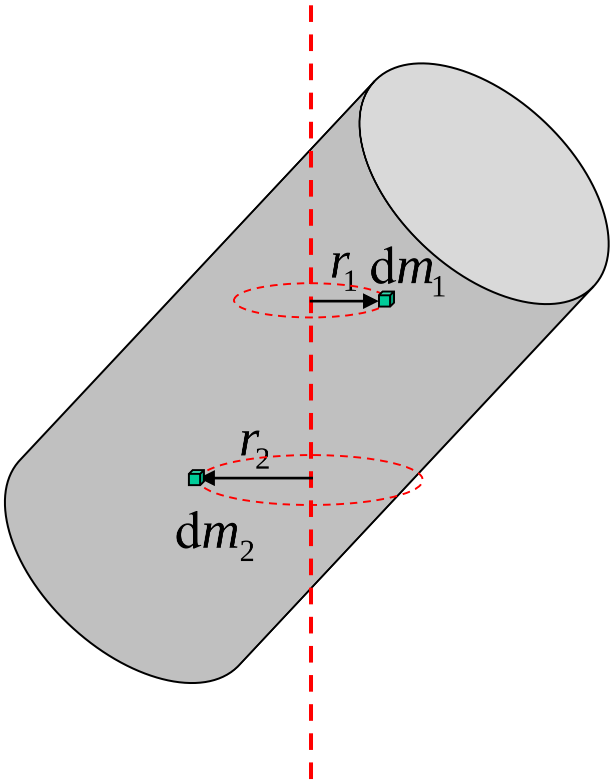
$$I = \sum_{i=1}^{\infty} \mathrm{d}m_i r_i^2$$

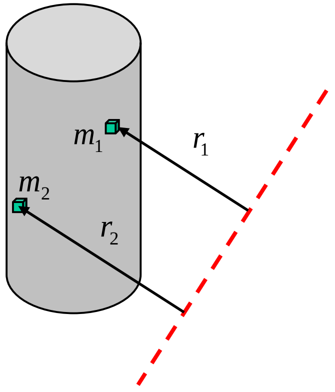
$$\Downarrow$$

$$I = \int r^2 \mathrm{d}m$$

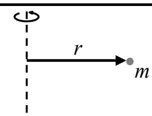
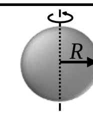
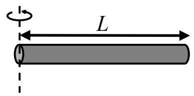
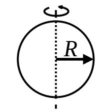
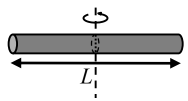
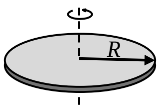
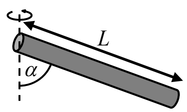
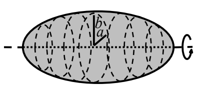
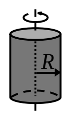
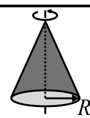
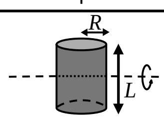
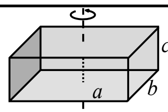
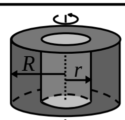
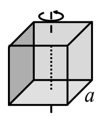
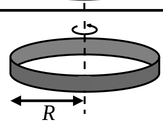
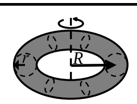
$$\Downarrow \quad (m = \rho V \Rightarrow \mathrm{d}m = \rho \mathrm{d}V)$$

$$I = \int r^2 \rho \mathrm{d}V$$





MOMENT OF INERTIA

Form		I	Form		I
Point		mr^2	**Ball**		$\frac{2}{5}mR^2$
Thin rod		$\frac{1}{3}mL^2$	**Sphere**		$\frac{2}{3}mR^2$
		$\frac{1}{12}mL^2$	**Disk**		$\frac{1}{2}mR^2$
		$\frac{1}{3}mL^2\sin^2\alpha$	**Ellipsoid**		$\frac{1}{5}m(b^2 + c^2)$
Cylinder		$\frac{1}{2}mR^2$	**Cone**		$\frac{3}{10}mR^2$
		$\frac{1}{12}m(L^2 + 3R^2)$	**Cuboid**		$\frac{1}{12}m(a^2 + b^2)$
Hollow cylinder		$\frac{1}{2}m(R^2 + r^2)$	**Cube**		$\frac{1}{6}ma^2$
Hoop		mR^2	**Torus**		$m\left(R^2 + \frac{3}{4}r^2\right)$

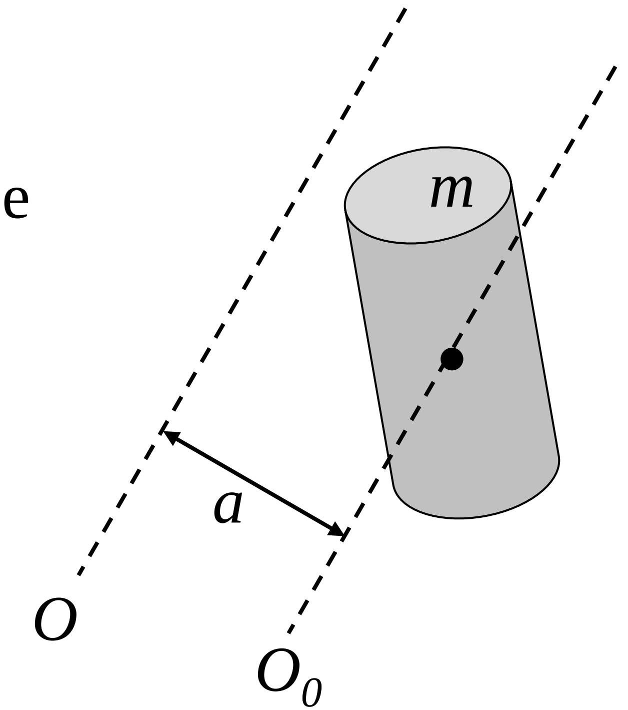
PARALLEL AXIS THEOREM
(Huygens–Steiner theorem, Steiner's theorem)

The moment of inertia I of a rigid body about any axis O is equal to the sume of

moment of inertia I_0 about a paralel axis O_0 through its center of mass

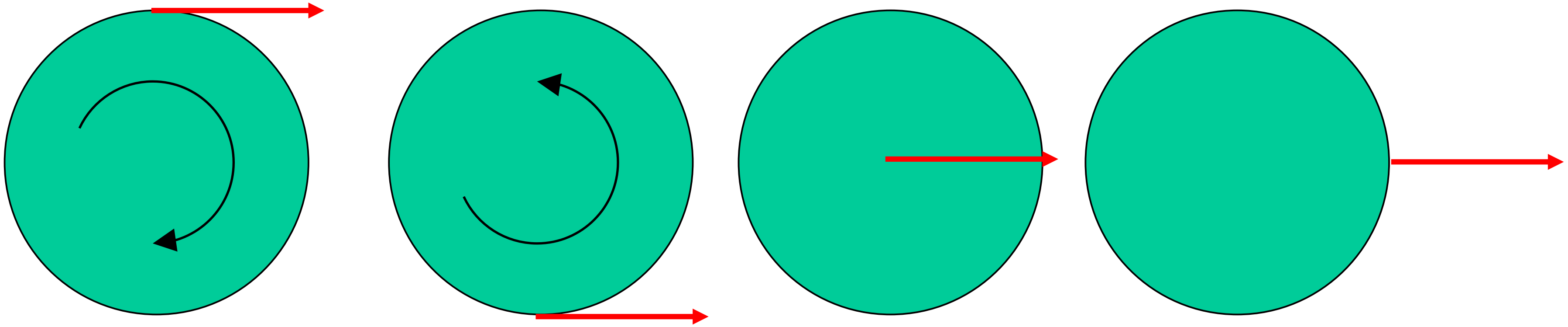
and

the product of the rigid body total mass m and the square of the perpendicular distance between the axes.



$$I = I_0 + ma^2$$

TORQUE



$$\vec{\tau} = \vec{r} \times \vec{F}$$

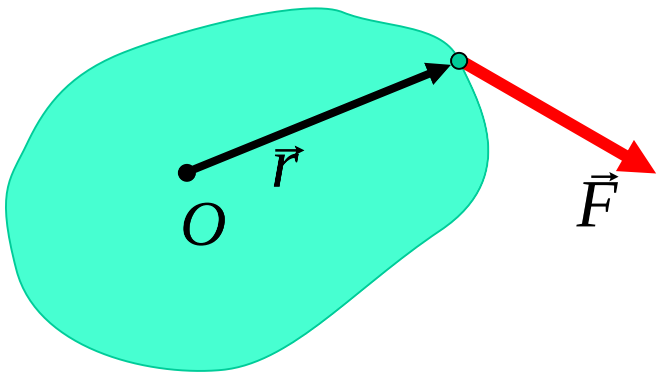
Torque $\vec{\tau}$ is defined as

the cross product

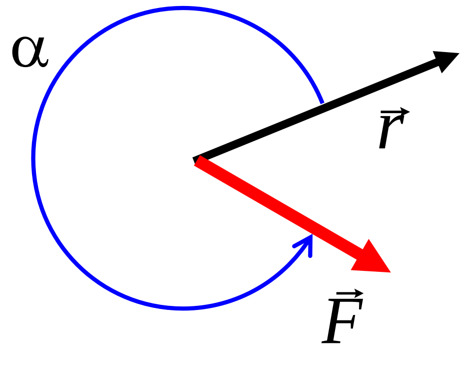
of the vector by which the force's application point is offset relative to the fixed suspension point (distance vector $\vec{r}$)

and the force vector $\vec{F}$, which tends to produce rotational motion.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

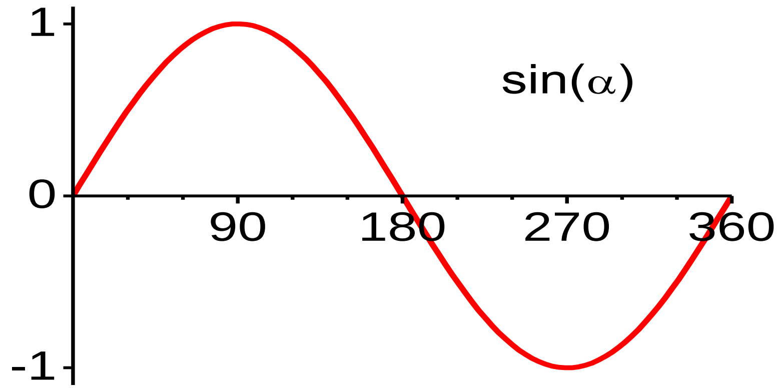


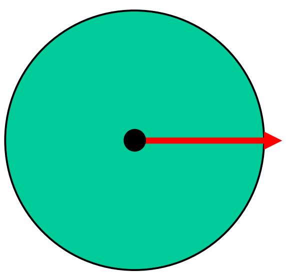
$$\tau = rF\sin(\alpha)$$

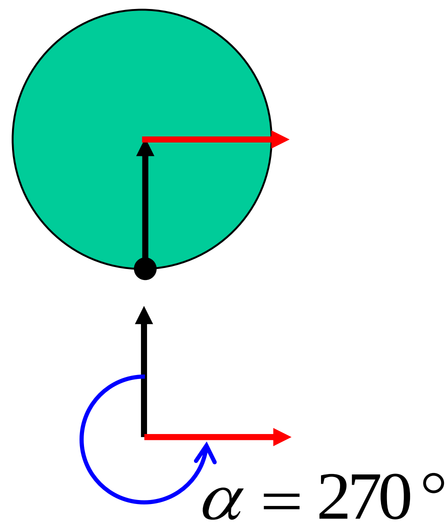


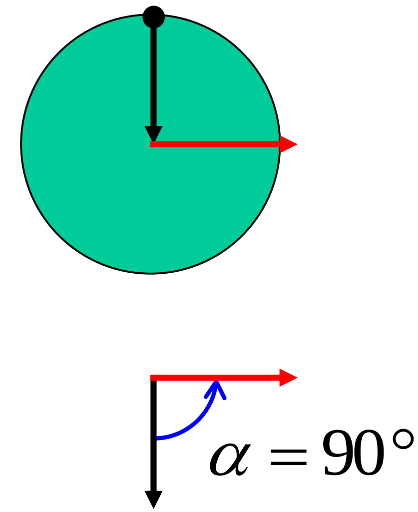
$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = rF\sin(\alpha)$$







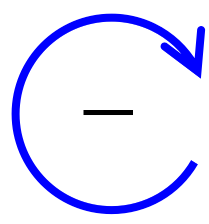


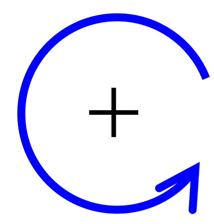
$r = 0$

$\tau = 0$

$\tau < 0$

$\tau > 0$

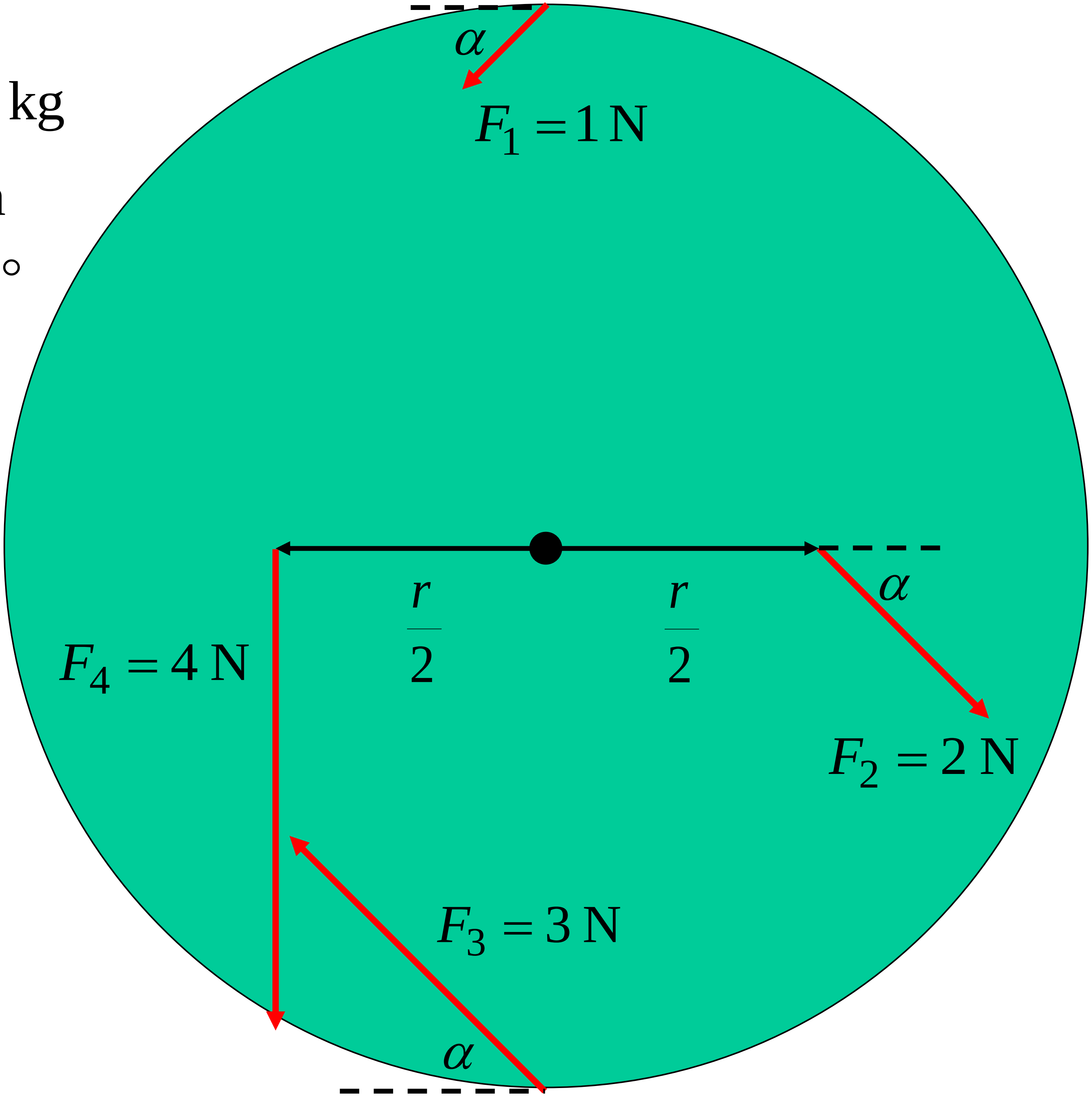




$m = 10\,\mathrm{kg}$

$r = 1\,\mathrm{m}$

$\alpha = 40°$

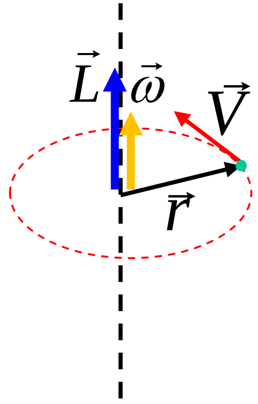


$a = ?$

$\varepsilon = ?$

ANGULAR MOMENTUM FOR A SINGLE PARTICLE

$$\vec{L} = \vec{r} \times \vec{p}$$



$$\vec{L} = \vec{r} \times m\vec{V}$$

$$L = rmV\sin(\alpha)$$

$$\vec{r} \perp \vec{V} \Rightarrow \alpha = 90^\circ$$

$$L = rmV$$

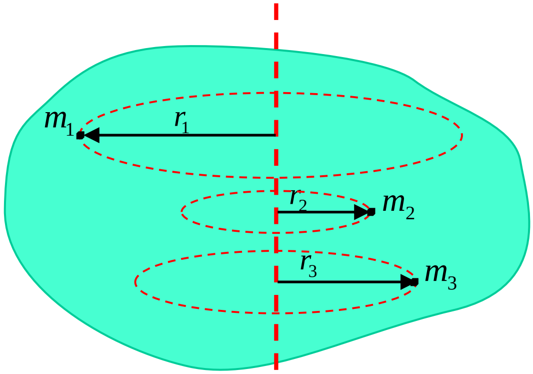
$$L = mr^2 \frac{V}{r} \Rightarrow \vec{L} = I_1\vec{\omega}$$

ANGULAR MOMENTUM OF THE RIGID BODY

For rigid body the total angular momentum is the vector sum of individual angular momentum of all points of the body:

$$\vec{L} = \sum_{i=1}^{\infty} \vec{L}_i$$

For each point the moment of inertia is individual, but the angular velocity is the same:



$$\vec{L} = \sum_{i=1}^{\infty} \left(I_i \vec{\omega} \right)$$

$$\vec{L} = \left(\sum_{i=1}^{\infty} I_i \right) \vec{\omega} \quad \Rightarrow \quad \vec{L} = I\vec{\omega}$$

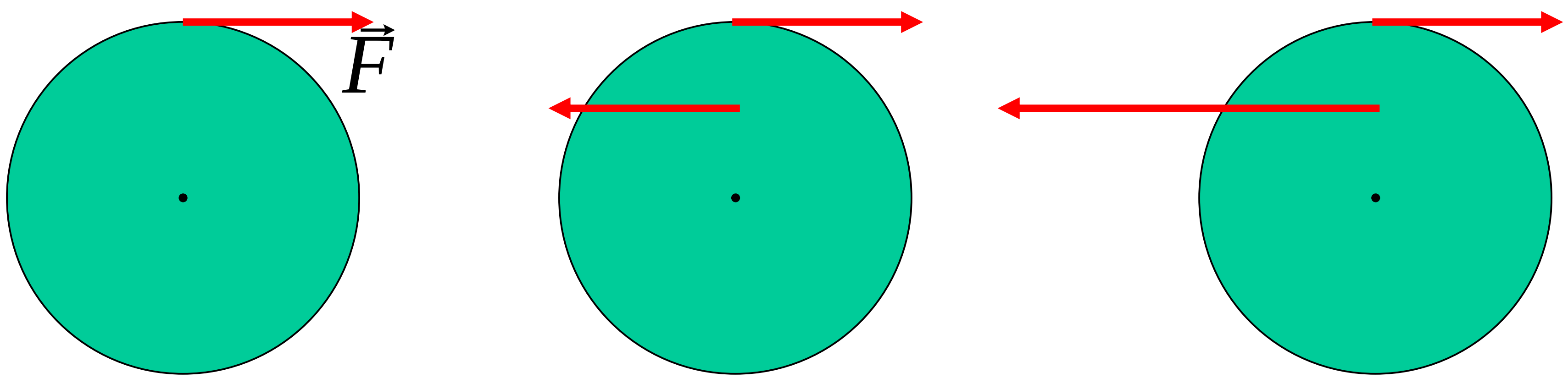
NEWTON'S FIRST LAW FOR ROTATION

$\vec{p} = \overrightarrow{\text{const}} \Leftrightarrow \vec{F}_w = \vec{0}$

$(\vec{a} = \vec{0})$

I : Rigid body stays at rest
or is at uniform cicular motion
if net torque is zero.

$$\vec{\varepsilon} = \vec{0}\left(\vec{\omega} = const\right) \Leftrightarrow \vec{\tau}_w = \vec{0}$$



$a > 0$ $\varepsilon < 0$

$a = 0$ $\varepsilon < 0$

$a < 0$ $\varepsilon = 0$

NEWTON'S SECOND LAW FOR ROTATION

$$\vec{F}_w = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

II : The rate of change of the angular momentum is equal to the net torque acting on the particle.

$$\vec{\tau}_w = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$$

$$\vec{F}_w = m\vec{a}$$

II : The net torque acting on a rigid body is equal to the product of the moment of inertia *I* i angular acceleration ε.

$$\vec{\tau}_w = I\vec{\varepsilon}$$

Momentum – product of mass and speed.

$$\vec{p} = m\vec{V}$$

Angular momentum – product of moment of inertia and angular velocity.

$$\vec{L} = I\vec{\omega}$$

NEWTON'S THIRD LAW FOR ROTATION

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

III : When one rigid body *A* exerts a torque on a second rigid body *B*, the second body simultaneously exerts a torque equal in magnitude and opposite in direction on the first rigid body.

$$\vec{\tau}_{AB} = -\vec{\tau}_{BA}$$

LINEAR vs ROTATIONAL MOTION

• position	r	$\Rightarrow$	α	• angular position
• velocity	V	$\Rightarrow$	ω	• angular velocity
• acceleration	a	$\Rightarrow$	ε	• angular acceleration
• inertia	m	$\Rightarrow$	I	• moment of inertia
• momentum	p	$\Rightarrow$	L	• angular momentum
• force	F	$\Rightarrow$	τ	• torque

WORK

$$W = \int \vec{F} \mathrm{d}\vec{r} \quad \Rightarrow \quad W = \int \vec{\tau} \mathrm{d}\vec{\alpha}$$

If

$$\vec{\tau} \parallel \mathrm{d}\vec{\alpha} \text{ and } \tau = \text{const}$$

then:

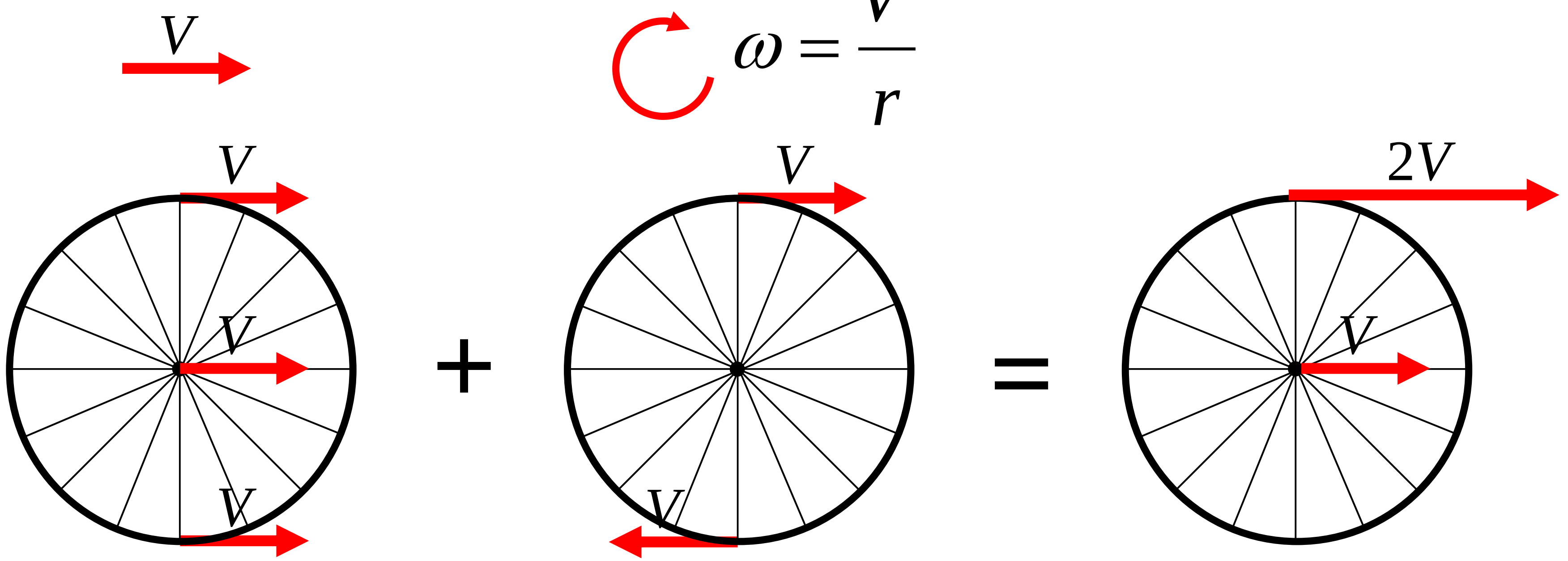
$$W = \pm \tau \cdot \alpha$$

KINETIC ENERGY

$$E_k = \frac{1}{2} m V^2 \quad \Rightarrow \quad E_k = \frac{1}{2} I \omega^2$$

ROLLING

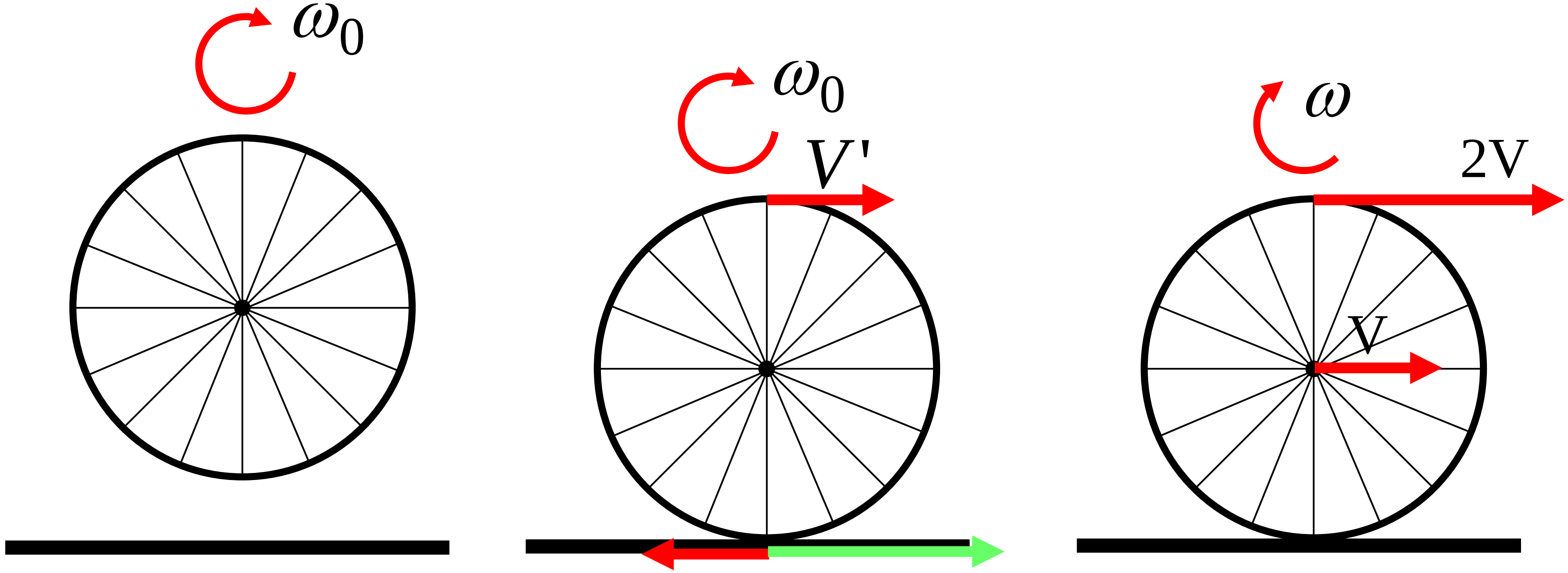
AS COMBINED ROTATION AND TRANSLATION | **AS PURE ROTATION**



$$\omega = \frac{V}{r}$$

$$E_k = E_{kp} + E_{ko} = \frac{1}{2} mV^2 + \frac{1}{2} I\omega^2$$

$$= \frac{1}{2}\left(mr^2 + I\right)\omega^2 = \frac{1}{2} I_1 \omega^2$$



$$V' = \omega_0 r \qquad T_k = \mu_k N$$

$$a = \frac{T_k}{m} > 0 \qquad \varepsilon = \frac{\tau_{T_k}}{I} = \frac{rT_k}{I} > 0 \qquad |\omega| = \frac{V}{r}$$

$$\omega_0 < 0 \qquad \omega = \omega_0 + \varepsilon t = \omega_0 + \frac{\tau_{T_k}}{I} t$$

$$(T_k = 0)$$

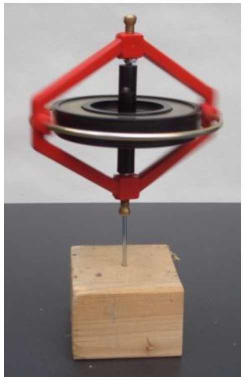
$$V_0 = 0 \qquad V = V_0 + at = \frac{T_k}{m} t$$

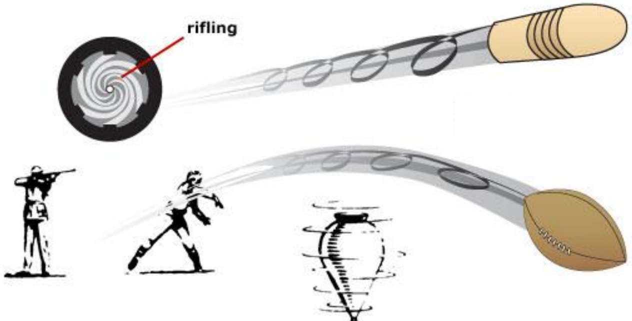
CONSERVATION OF ANGULAR MOMENTUM

$$\vec{\tau}_{wex} = \vec{0} \quad \Rightarrow \quad \frac{d(\vec{L}_w)}{dt} = \vec{0} \quad \Rightarrow \quad \vec{L}_w = \text{const}$$

If the net external torque $\vec{\tau}_{we}$ acting on a system is zero, the net angular momentum of the system remains constant, no matter what changes take place within the system.

$$\vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \ldots + \vec{L}_n = \text{const}$$





CONSERVATION LAW

- Conservation of mechanical energy

$$\vec{F}_w = \vec{F}_p \Rightarrow E_c = K + U = \text{const}$$

- Conservation of total energy

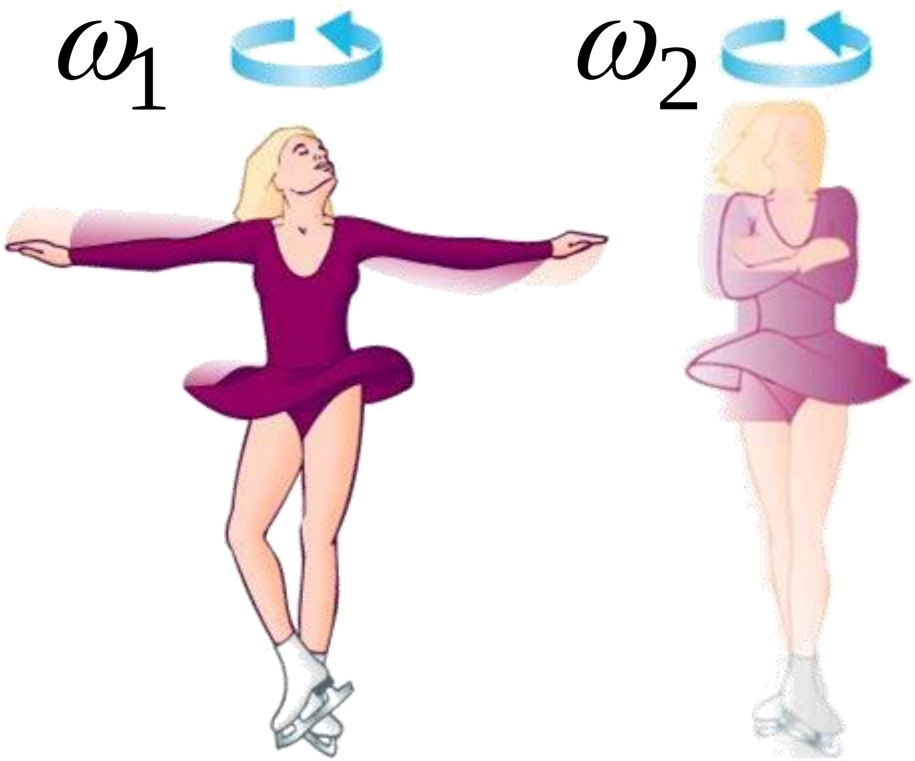
$$\Delta K + \Delta U = W_{\vec{F}_{ex}} - \Delta U_{\text{int}}$$

- Conservation of momentum

$$\vec{F}_{wex} = \vec{0} \Rightarrow \vec{p}_c = \sum \vec{p}_i = \overrightarrow{\text{const}}$$

- Conservation of angular momentum

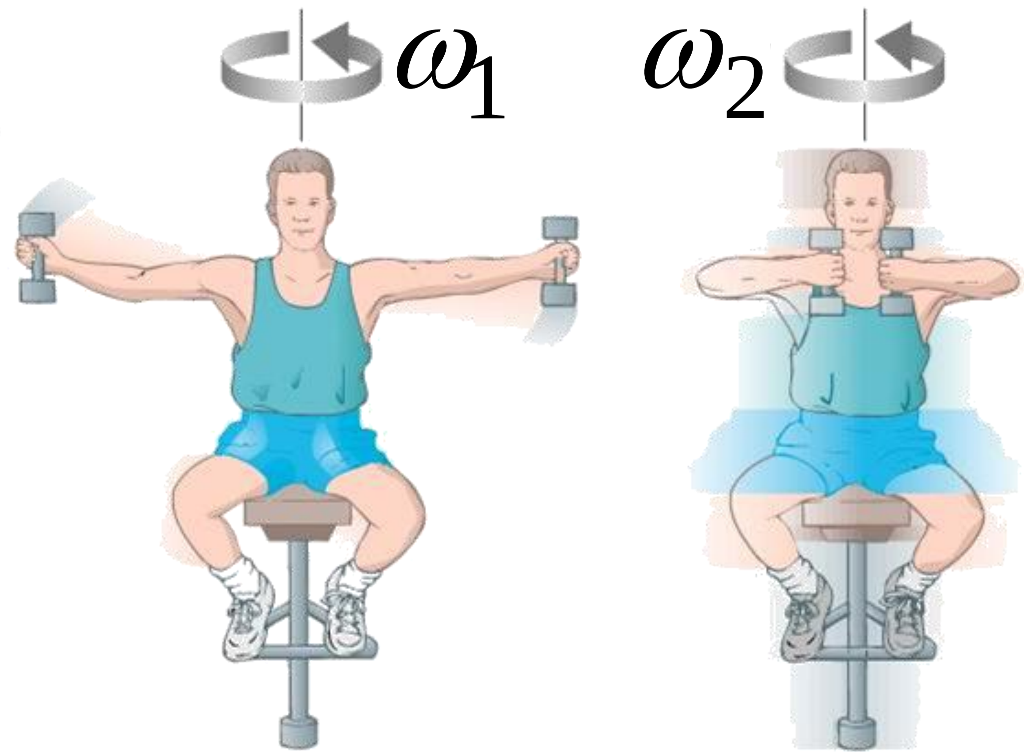
$$\vec{\tau}_{wex} = \vec{0} \Rightarrow \vec{L} = \sum \vec{L}_i = \overrightarrow{\text{const}}$$



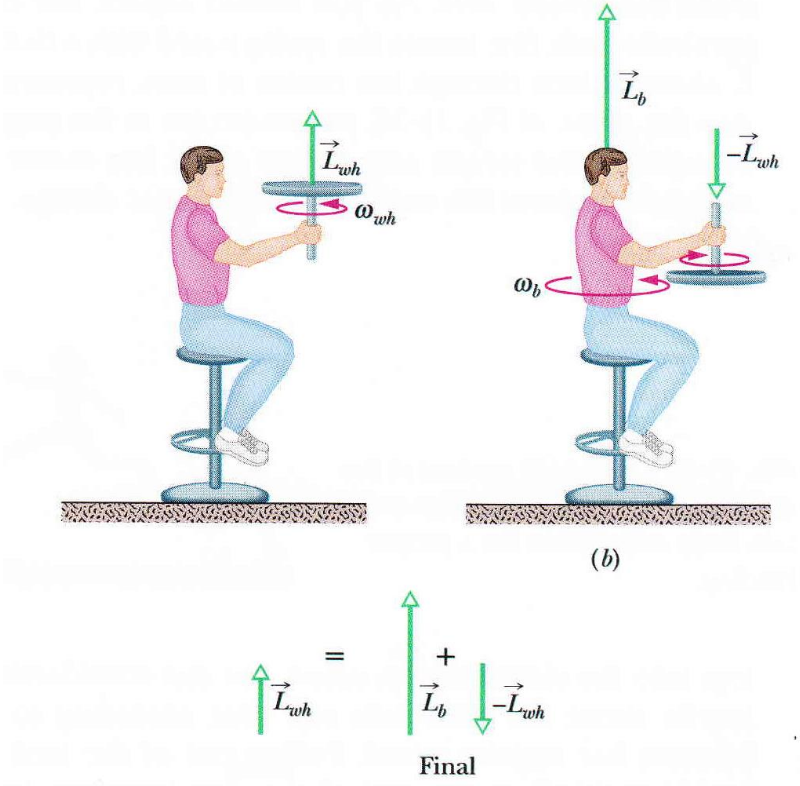
$$\vec{\tau}_{ex} = \vec{0} \Rightarrow L_1 = L_2$$

$$I_1\omega_1 = I_2\omega_2$$

$$I_1 > I_2 \Rightarrow \omega_1 < \omega_2$$



A change in the mass distribution causes a change in the angular velocity.


$\vec{L}_{wh}$
ω_{wh}
$\vec{L}_b$
$-\vec{L}_{wh}$
ω_b
(b)
$\vec{L}_{wh} = \vec{L}_b + -\vec{L}_{wh}$
Final