Kinematics

Kinematics is a branch of classical mechanics that describes the motion of points, bodies and systems of bodies without considering the mass of each or the forces that caused the motion

Reference frame (a frame of reference) consists of an abstract coordinate system and the set of physical reference points that uniquely locate and orient the coordinate system

Motion is a change in position of an object over time

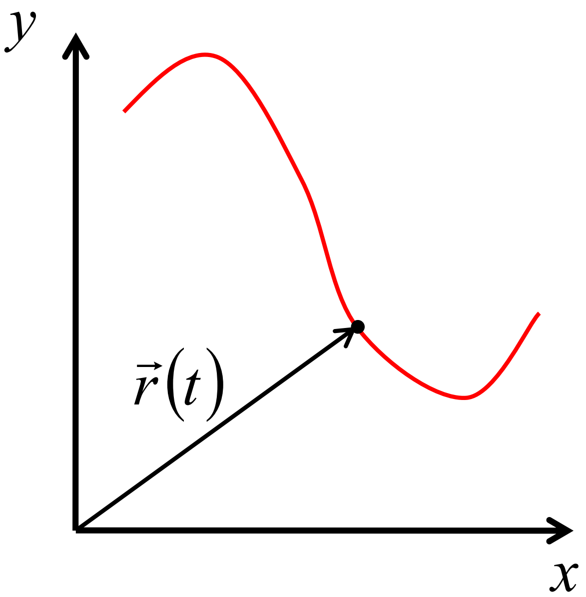
Depending on the chosen reference frame the object could be in motion or in rest. As there is no absolute frame of reference, absolute motion cannot be determined.

A trajectory or flight path is the path that a moving object follows through space as a function of time.

Position vector

Position vector defines the location of a particle relative to the origin of a coordinate system at some time *t*, its position on a trajectory.

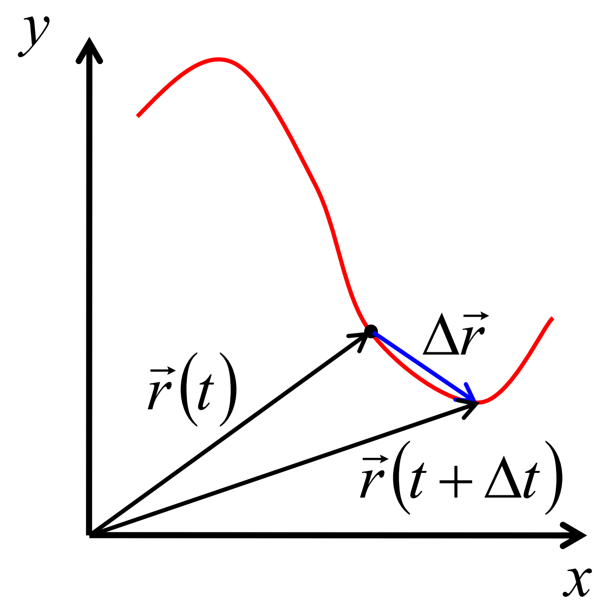
$$\vec{r}(t)$$



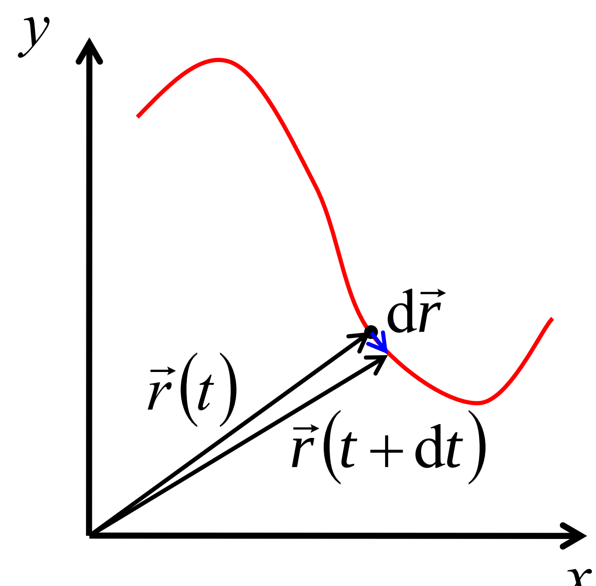
Displacement vector

If a particle moves so that its position vector changes in time interval Δt than the particle displacement is:

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$$



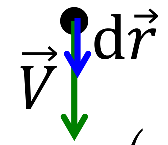
When $\Delta t \rightarrow 0$ displacement becomes infinitesimally small:



Velocity (instantaneous velocity vector)

Velocity $\vec{V}$:

– describe the displacement direction: $\mathrm{d}\vec{r} \,||\, \vec{V}$



$\left(\mathrm{d}\vec{r} = \vec{V}\mathrm{d}t\right)$

– describe the rate of change of the position in time:

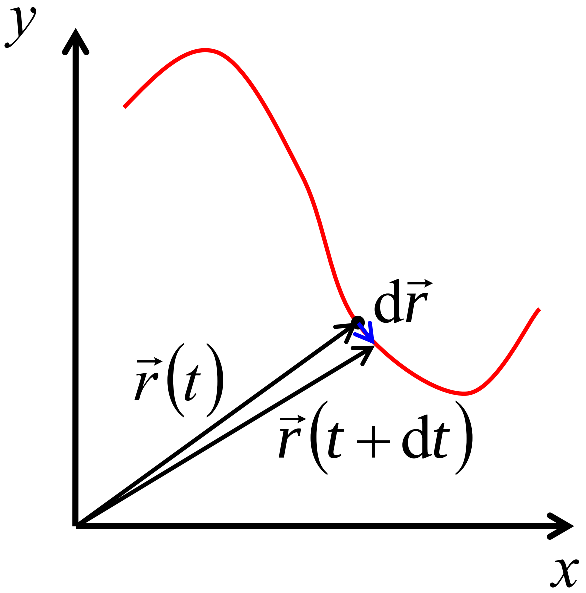
$$\vec{V} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \dot{\vec{r}} = \vec{r}\,'$$

Velocity (instantaneous velocity vector) is a derivative of the position vector with respect to the time.

The velocity vector value (magnitude) is called speed.

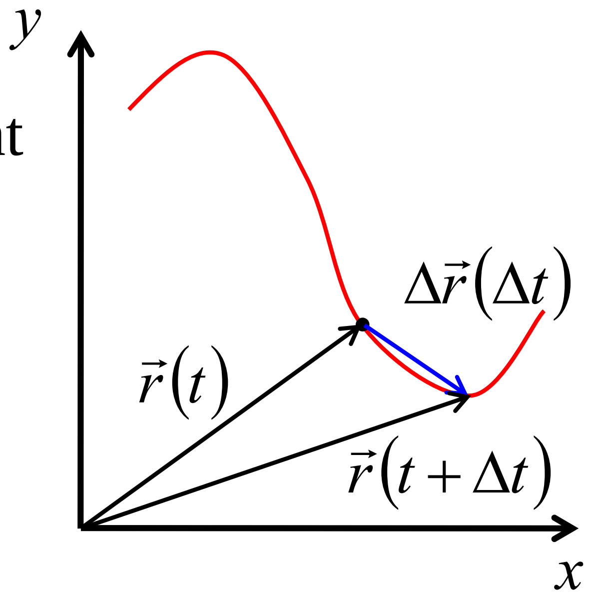
Instantaneous and average velocity

$$\vec{V} = \frac{d\vec{r}}{dt}$$



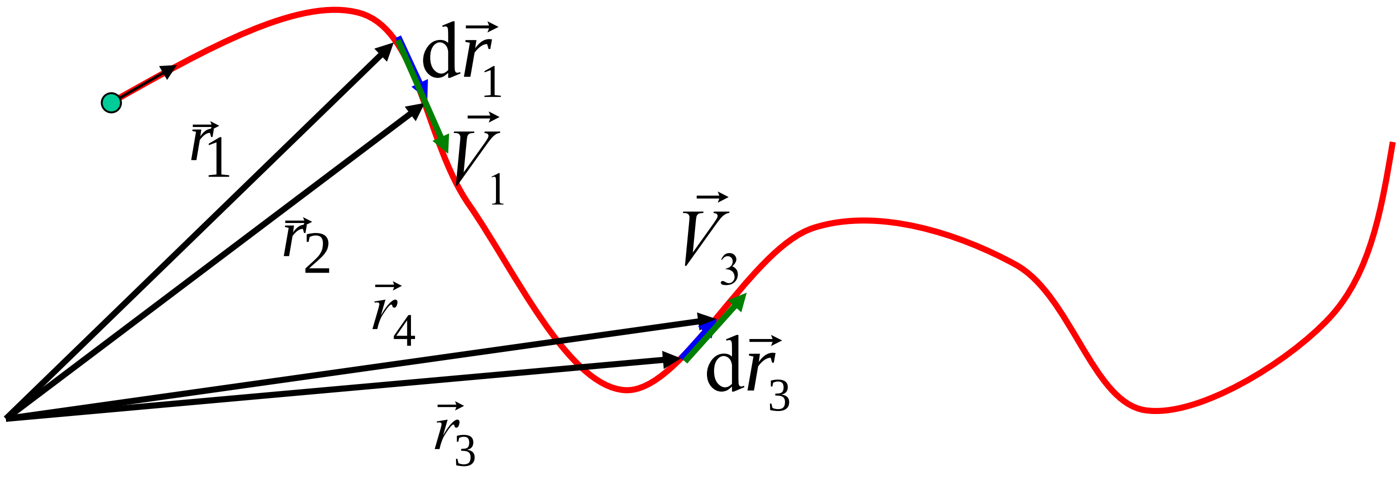
<u>Average velocity</u> is the ratio of the displacement of a particle divided by the time interval.

$$\vec{V}_{śr} = \frac{\Delta\vec{r}}{\Delta t}$$



The velocity $\vec{V}$ direction is always the same as the direction of the displacement vector $\Delta\vec{r}$:

$$\vec{V} = \frac{1}{\mathrm{d}t}\mathrm{d}\vec{r}$$



The velocity is always tangent to the path of the particle – its trajectory.

The velocity is a derivative of the position vector with respect to the time.

$$\vec{V} = \frac{d\vec{r}}{dt}$$

The position vector is an integral of the velocity with respect to the time.

$$\vec{r} = \int \vec{V} dt$$

Acceleration (instantaneous acceleration)

Velocity $\vec{V}$:

– describe the rate of change of the position in time:

$$\vec{V} = \frac{d\vec{r}}{dt}$$

The rate of change of the velocity in time is described by acceleration:

$$\vec{a} = \frac{d\vec{V}}{dt}$$

Acceleration vector (instantaneous acceleration vector) is a derivative of the velocity vector with respect to the time.

The acceleration is a derivative of the velocity with respect to the time.

$$\vec{a} = \frac{d\vec{V}}{dt}$$

$$\vec{V} = \int \vec{a}\,dt$$

The velocity is an integral of the acceleration with respect to the time.

The acceleration is a derivative of the velocity with respect to the time.

$$\vec{a} = \frac{\mathrm{d}\vec{V}}{\mathrm{d}t} = \dot{\vec{V}} = \vec{V}'$$

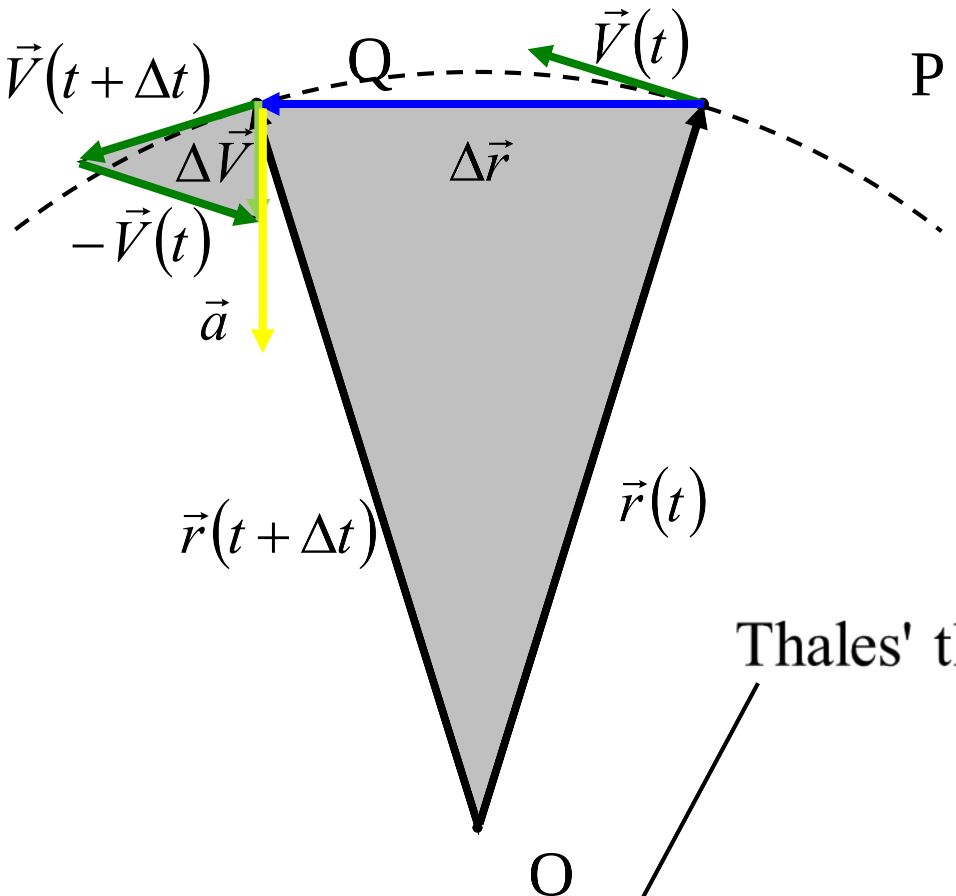
$$\vec{a} = \frac{\mathrm{d}\vec{V}}{\mathrm{d}t} = \frac{\mathrm{d}\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \ddot{\vec{r}} = \vec{r}''$$

The acceleration is a second derivative of the position with respect to the time.

Uniform circular motion

– the body moves in a circular path with uniform speed: $r = \text{const}$

$$V = \text{const}$$



$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$$

$$\Delta\vec{V} = \vec{V}(t+\Delta t) - \vec{V}(t)$$

$$\vec{a} = \frac{1}{\Delta t}\Delta\vec{V}$$

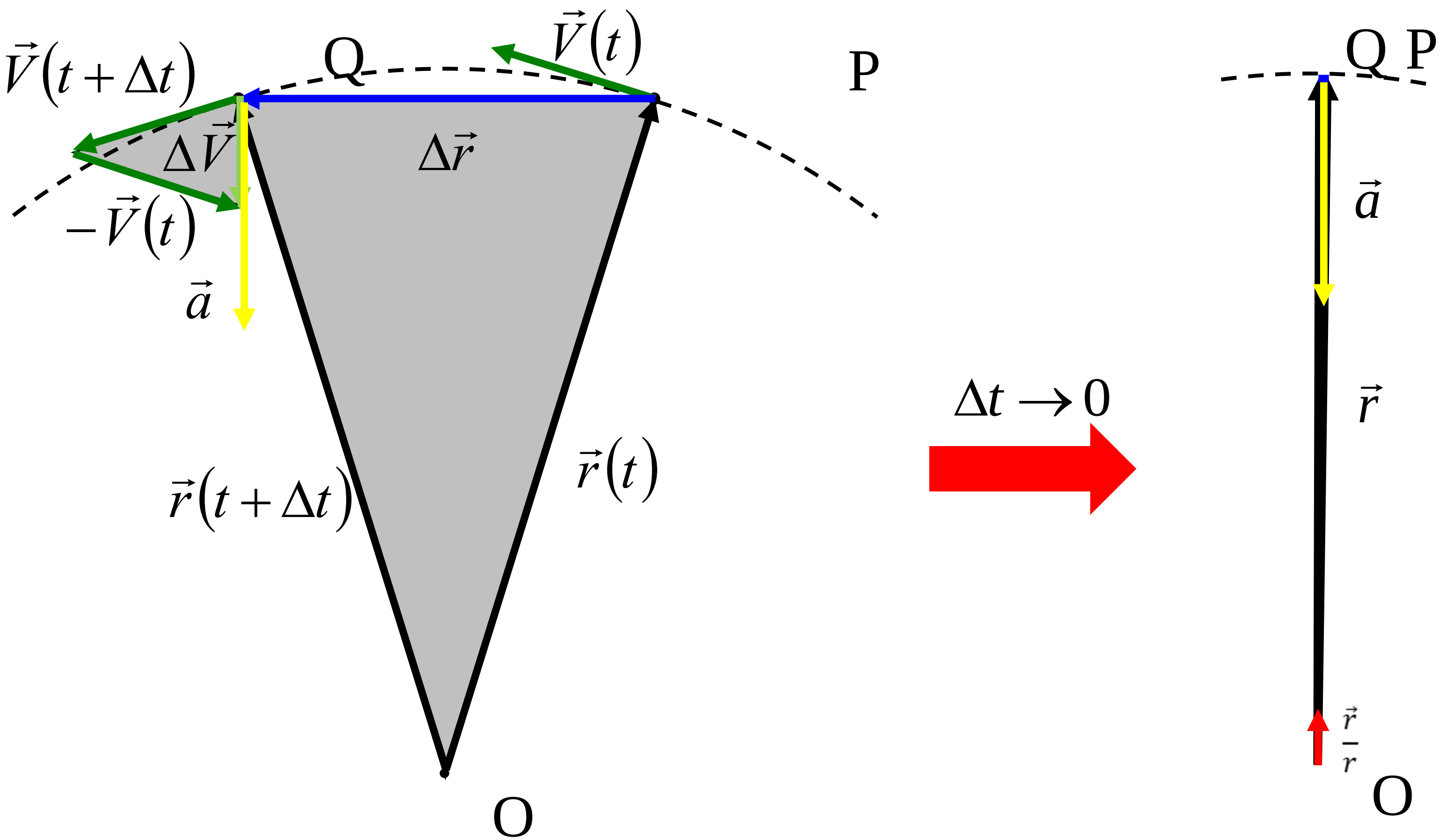
Thales' theorem: $\dfrac{\Delta V}{\Delta r} = \dfrac{V}{r}$

$$\lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{\mathrm{d}r}{\mathrm{d}t} = V$$

$$a = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta r}\frac{\Delta r}{\Delta t} = \lim_{\Delta t \to 0} \frac{V}{r}\frac{\Delta r}{\Delta t} = \frac{V}{r}\lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{V^2}{r}$$

Uniform circular motion

– the body moves in a circular path with uniform speed

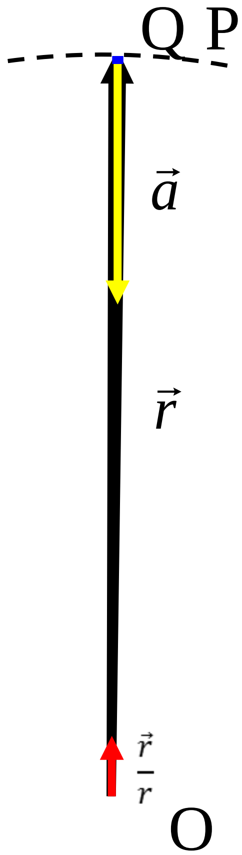


1) $\vec{a}$ is parallel to the unit vector $\frac{\vec{r}}{r}$: $\vec{a} \propto \frac{\vec{r}}{r}$

2) $\vec{a}$ has the opposite direction to $\frac{\vec{r}}{r}$: $\vec{a} \propto -\frac{\vec{r}}{r}$

Uniform circular motion

– the body moves in a circular path with uniform speed



1) $\vec{a}$ is parallel to the unit vector $\frac{\vec{r}}{r}$: $\vec{a} \propto \frac{\vec{r}}{r}$

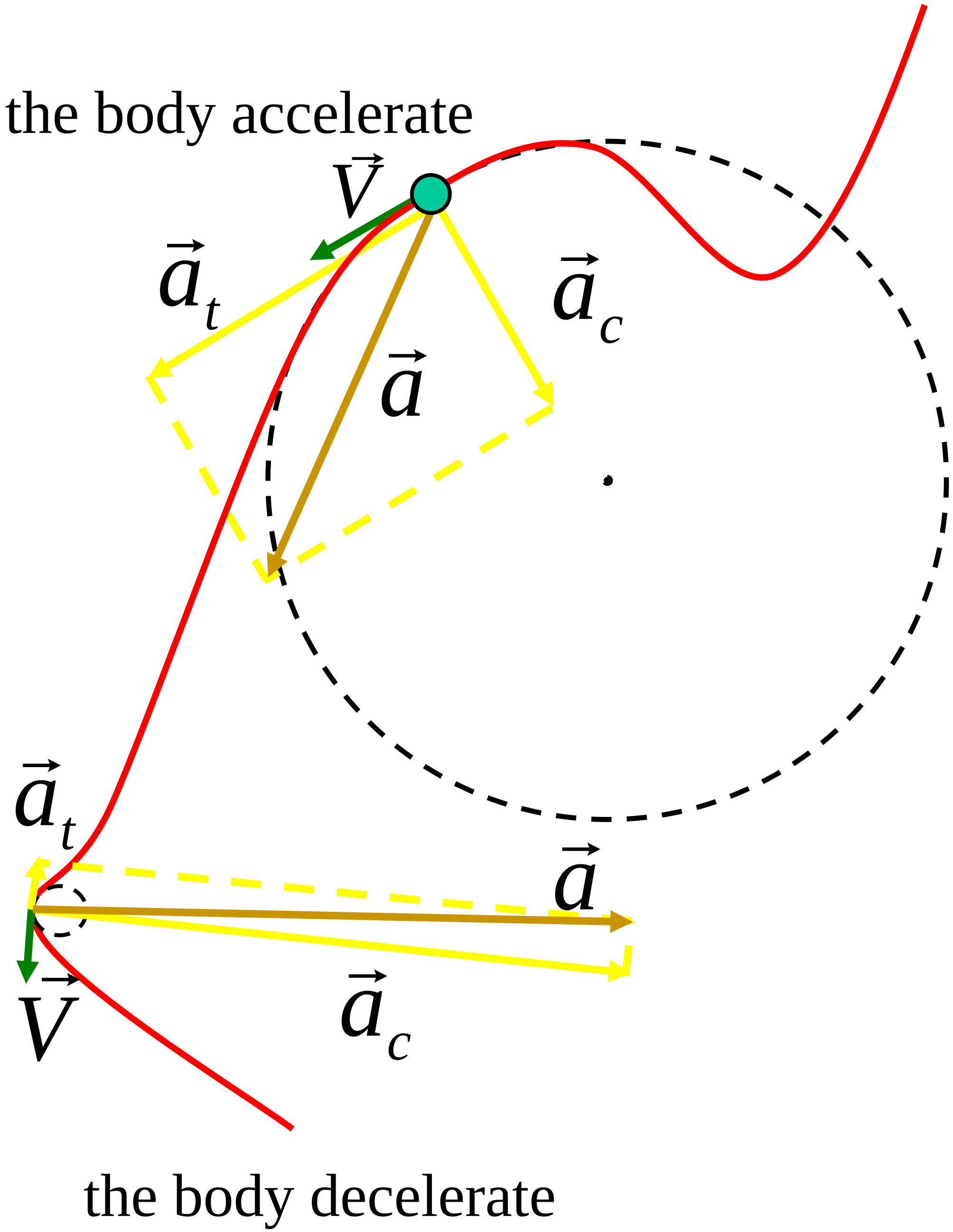
2) $\vec{a}$ has the opposite direction to $\frac{\vec{r}}{r}$: $\vec{a} \propto -\frac{\vec{r}}{r}$

– is directed to the centre of the circle

3) $\vec{a}$ has a value $\frac{v^2}{r}$

$$\vec{a}_c = -\frac{V^2}{r}\frac{\vec{r}}{r}$$

The centripetal acceleration in circular motion (normal or radial acceleration)



$\vec{a}_c$ – the centripetal acceleratio

$$\vec{a}_c = -\frac{V^2}{r}\frac{\vec{r}}{r}$$

$\vec{a}_t$ – the tangential acceleration

$$\vec{a}_t = \frac{\mathrm{d}V}{\mathrm{d}t}\frac{\vec{V}}{V}$$

$\vec{a}$ – the total acceleration

$$\vec{a} = \vec{a}_t + \vec{a}_c$$

Example 1

$$\vec{r}(t)=\vec{r}_0+\vec{V}_0 t+\tfrac{1}{2}\vec{a}t^2 \qquad \vec{r}_0,\ \vec{V}_0,\ \vec{a} \text{ - const}$$

$$\vec{V}(t)=\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}=\frac{\mathrm{d}\left(\vec{r}_0+\vec{V}_0 t+\tfrac{1}{2}\vec{a}t^2\right)}{\mathrm{d}t}=\frac{\mathrm{d}\vec{r}_0}{\mathrm{d}t}+\frac{\mathrm{d}\left(\vec{V}_0 t\right)}{\mathrm{d}t}+\frac{\mathrm{d}\left(\tfrac{1}{2}\vec{a}t^2\right)}{\mathrm{d}t}$$

$=0 \qquad =\vec{V}_0 \qquad =\vec{a}t$

$$\vec{V}(t)=\vec{V}_0+\vec{a}t$$

$$\vec{a}(t)=\frac{\mathrm{d}\vec{V}}{\mathrm{d}t}=\frac{\mathrm{d}\left(\vec{V}_0+\vec{a}t\right)}{\mathrm{d}t}=\frac{\mathrm{d}\left(\vec{V}_0\right)}{\mathrm{d}t}+\frac{\mathrm{d}\left(\vec{a}t\right)}{\mathrm{d}t}$$

$=0 \qquad =\vec{a}$

$$\vec{a}(t)=\vec{a}=\overrightarrow{\mathrm{const}}$$

The equations of motion with constant acceleration

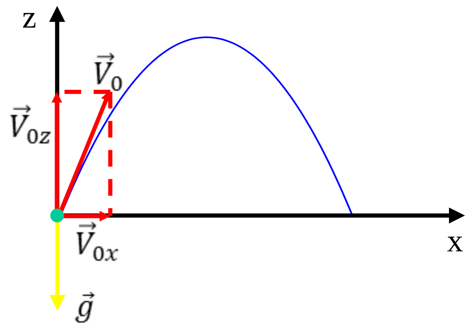
$$\vec{r}(t) = \vec{r}_0 + \vec{V}_0 t + \frac{1}{2}\vec{a}t^2$$

$$\vec{V}(t) = \vec{V}_0 + \vec{a}t$$

$$\vec{a} = \overrightarrow{\text{const}}$$

- when an object is speeding up – acceleration: $\frac{\vec{a}}{a} = \frac{\vec{V}}{V}$
- when an object is speeding down – deceleration: $\frac{\vec{a}}{a} = -\frac{\vec{V}}{V}$

Ballistic trajectories (Projectile motion)



$$\vec{r}(t) = \vec{r}_0 + \vec{V}_0 t + \tfrac{1}{2}\vec{a}t^2$$
$$\vec{V}(t) = \vec{V}_0 + \vec{a}t$$
$$\vec{a} = \overrightarrow{\text{const}}$$

$$\vec{r}_0 = (0,0,0) \quad \vec{V}_0 = (V_{0x}, 0, V_{0z}) \quad \vec{a} = (0,0,-g)$$

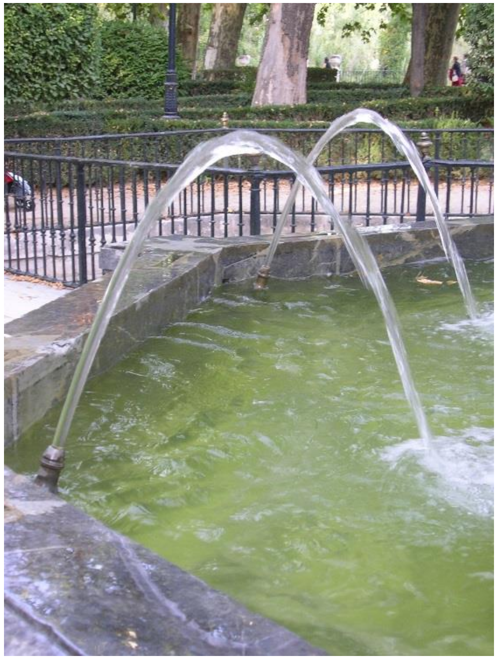
$$\begin{cases} x = V_{0x}t \\ V_x = V_{0x} \\ a_x = 0 \end{cases} \qquad \begin{cases} y = 0 \\ V_y = 0 \\ a_y = 0 \end{cases} \qquad \begin{cases} z = V_{0z}t - \tfrac{1}{2}gt^2 \\ V_z = V_{0z} - gt \\ a_z = -g \end{cases}$$

- along the x axis – uniform motion
- along the y axis – the body is at rest
- along the z axis – initial deceleration, than acceleration

Ballistic trajectories (Projectile motion)

$$\begin{cases} x = V_{0x} t \\ V_x = V_{0x} \\ a_x = 0 \end{cases} \qquad \begin{cases} y = 0 \\ V_y = 0 \\ a_y = 0 \end{cases} \qquad \begin{cases} z = V_{0z} t - \frac{1}{2} g t^2 \\ V_z = V_{0z} - g t \\ a_z = -g \end{cases}$$

- Time of flight
- Maximum height of projectile
- Horizontal range
- Maximum distance of projectile
- Angle θ to hit coordinate (x,z)
- Trajectory – parabolic



$$t = \frac{x}{V_{0x}} \Rightarrow z = \frac{V_{0z}}{V_{0x}} x - \frac{g}{2V_{ox}^2} x^2$$

Uniform linear motion

$$\begin{cases} \vec{r}(t) = \vec{r}_0 + \vec{V}_0 t + \frac{1}{2}\vec{a}t^2 \\ \vec{V}(t) = \vec{V}_0 + \vec{a}t \\ \vec{a}(t) = \text{const} \end{cases}$$

Uniform circular motion

$$\begin{cases} \vec{\alpha}(t) = \vec{\alpha}_0 + \vec{\omega}_0 t + \frac{1}{2}\vec{\varepsilon}t^2 \\ \vec{\omega}(t) = \vec{\omega}_0 + \vec{\varepsilon}t \\ \vec{\varepsilon}(t) = \text{const} \end{cases}$$

α – the angle of rotation

ω – the angular velocity

ε – the angular acceleration

Example 2

$$x(t) = A\cos(\omega_0 t) \qquad A,\ \omega_0 - \text{const}$$

$$V(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}(A\cos(\omega_0 t))}{\mathrm{d}t} = A\frac{\mathrm{d}(\cos(\omega_0 t))}{\mathrm{d}t}$$

$$V(t) = -A\omega_0 \sin(\omega_0 t)$$

$$a(t) = \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}(-A\omega_0 \sin(\omega_0 t))}{\mathrm{d}t} = -A\omega_0 \frac{\mathrm{d}(\sin(\omega_0 t))}{\mathrm{d}t}$$

$$a(t) = -A\omega_0^2 \cos(\omega_0 t)$$

$$a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \qquad A\cos(\omega_0 t) = x(t)$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega_0^2 x$$

$$\frac{d^2x}{dt^2} = -\omega_0^2 x$$

– when the acceleration is directly proportional to the displacement but in opposite direction

then the displacement varies according to the equation

$$x(t) = A\cos(\omega_0 t)$$

It is **simple harmonic motion** with amplitud A and period

$$T = \frac{2\pi}{\omega_0}$$